THE HYDRODYNAMICS OF AN OSCILLATING DISK IN A ROTATING FLUID

(K GIDRODINAMIKE KOLEBANII DISKA VO VRASHCHAIUSHCHEISIA ZHIDKOSTI)

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This problem arose in connection with the study of the properties of helium II under rotating conditions. A disk under torsional oscillations about its axis is known to yield valuable experimental information about the properties of liquid helium [1,2]. In this connection the corresponding hydrodynamic problem for a stationary fluid was solved [3]. Latterly the oscillating disk method has been employed for studying helium in rotation [4,5,6] and it has revealed special features peculiar to a quantum fluid [7]. These special features are due to the behavior of super-fluid components in helium II when, at the same time, the normal component of helium behaves like a normal liquid. When interpreting experimental data it is essential to distinguish between the quantum effects of motion of super-fluid components and their interaction with the normal ones and the effects of classical motion of normal components. This is the reason why we study here the problem of torsional oscillations of a disk about its own axis in a conventional rotating fluid.

1. Formulation of the problem. Let an infinite incompressible fluid of density ρ rotate at constant angular velocity ω_0. A circular disk of radius R and thickness h, the axis of which coincides with that of the rotating fluid, rotates with it, and on that motion, oscillations about the same axis at frequency Ω are superposed. We can, therefore, express the disk motion in cylindrical coordinates as follows

$$\varphi = \omega_0 t + \varphi_0 e^{i\Omega t} \tag{1.1}$$

We look for a solution of the hydrodynamic equations (Navier-Stokes equation and equation of continuity) in this form

$$v_r = u_r(r, z)\, e^{i\Omega t}, \qquad v_\varphi = u_\varphi(r, z)\, e^{i\Omega t} + \omega_0 r, \qquad v_z = w_z(z)\, e^{i\Omega t} \tag{1.2}$$

$$p = p_0 + \frac{1}{2}\rho\omega_0{}^2 r^2 + p_1(z)\, e^{i\Omega t} \tag{1.3}$$

Here p_0 is the constant pressure (pressure at axis with no oscillation). In the Equations (1.2) and (1.3) terms containing functions u, w and p_1 describe the perturbations of the well known solution to this system for the case of pure rotation, which arise as a result of the oscillations.

Let us limit ourselves to the case of small amplitudes of oscillation and we will linearise the system of hydrodynamic equations as applied to the oscillation perturbations of velocity and pressure;

$$i\Omega u_r - 2\omega_0 u_\varphi = \nu \frac{\partial^2 u_r}{\partial z^2} + \nu \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \tag{1.4}$$

$$i\Omega u_\varphi + 2\omega_0 u_r = \nu \frac{\partial^2 u_\varphi}{dz^2} + \nu \frac{d}{dr}\frac{1}{r}\frac{d}{dr}(r u_\varphi) \tag{1.5}$$

$$i\Omega w_z = -\frac{1}{\rho}\frac{dp_1}{\partial z} + \nu \frac{d^2 w_z}{\partial z^2}, \qquad \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{dw_z}{\partial z} = 0 \tag{1.6}$$

Here $\nu = \eta/\rho$, the kinematic viscosity of the fluid.

To find the functions u_ϕ and u_r we use Equations (1.4) and (1.5) which, when combined, yield

$$\left[i(\Omega \mp 2\omega_0) - \nu \frac{\partial^2}{\partial z^2}\right](u_\varphi \pm i u_r) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)(u_\varphi \pm i u_r) \tag{1.7}$$

2. Solution of the system. To solve the system (1.7) we use a method which was employed by Mariens and Van Paemel for solving the similar problem in a stationary fluid (8). Let us divide the space occupied by the fluid into three regions, as shown in Fig. 1 and we will study the motion in each of them separately. Clearly when $h \ll R$ and with small depth of penetration the most important region will be 3. Regions 1 and 2 are only incorporated in the discussion for approximate calculation of corrections to the case of an infinite disk. (Owing to the symmetry of the problem, the regions symmetrical to 2 and 3 are not studied separately).

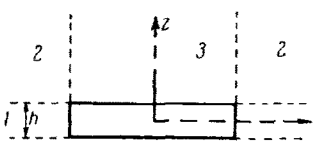

Fig. 1.

Region 1. $(R \leqslant r < \infty,\ -{}^1/_2 h \leqslant z \leqslant {}^1/_2 h.)$. (1)

Assuming h to be small we can neglect change of velocity with z in this region. Then Equations (1.7) take the form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k_{\mp}{}^2-\frac{1}{r^2}\right)(u_\varphi \pm iu_r)=0,\ k_{\mp}^2=-i\,\frac{\Omega\mp 2\omega_0}{\nu} \tag{2.1}$$

In terms of formulas (1.1) and (1.2), the boundary conditions on the disk surfaces are of the form

$$u_r^{(1)}(R)=0, \qquad u_\varphi^{(1)}(R)=i\Omega\varphi_0 R_i \text{ for } -\tfrac{1}{2}h\leqslant z\leqslant\tfrac{1}{2}h \tag{2.2}$$

At infinity we have

$$u_r^{(1)}(\infty)=u_\varphi^{(1)}(\infty)=0 \tag{2.3}$$

With these boundary conditions we arrive at the following expressions for the solution of equation (2.1) in terms of Hankel functions of the first kind of first order $H_1^{(2)}(\zeta)$;

$$u_r^{(1)}(r)=-\frac{\Omega\varphi_0 R}{2}\left[\frac{H_1^{(1)}(k_+r)}{H_1^{(1)}(k_+R)}-\frac{H_1^{(1)}(k_-r)}{H_1^{(1)}(k_-R)}\right]$$
$$u_\varphi^{(1)}(r)=i\frac{\Omega\varphi_0 R}{2}\left[\frac{H_1^{(1)}(k_+r)}{H_1^{(1)}(k_+R)}+\frac{H_1^{(1)}(k_-r)}{H_1^{(1)}(k_-R)}\right] \tag{2.4}$$

where

$$\operatorname{Im}(k_\pm)>0 \tag{2.5}$$

Region 2. In this region ($R < r < \infty$, $z > 1/2\ h$) we search for the solution in the following form

$$u_\varphi+iu_r=A(r)\,w_+(z), \qquad u_\varphi-iu_r=B(r)\,w_-(z) \tag{2.6}$$

where $A(r)$ and $B(r)$ satisfy Equations (2.1) and conditions of type (2.3) for $r=\infty$.

In other words a similar radial fluid velocity distribution is presupposed in regions 1 and 2 and this permits solutions (1) and (2) to be joined over the surface where they meet. The matching conditions, replacing the absent boundary conditions of region 2 by boundary conditions on the disk surface, are of this form

$$u_r^{(2)}\left(r,\ \tfrac{1}{2}h\right)=u_r^{(1)}(r), \quad u_\varphi^{(2)}\left(r,\ \tfrac{1}{2}h\right)=u_\varphi^{(1)}(r) \quad \text{for } r\geqslant R \tag{2.7}$$

Bearing in mind also conditions at infinity

$$u_r^{(2)}(r,\infty)=u_\varphi^{(2)}(r,\infty)=0 \tag{2.8}$$

and, condition (2.5), we arrive at

$$u_r^{(2)}(r,z) = -\frac{\Omega\varphi_0 R}{2}\left\{\frac{H_1^{(1)}(k_+r)}{H_1^{(1)}(k_+R)}\exp\left[ik_+\left(z-\frac{h}{2}\right)\right] - \frac{H_1^{(1)}(k_-r)}{H_1^{(1)}(k_-R)}\exp\left[ik_-\left(z-\frac{h}{2}\right)\right]\right\}$$

$$u_\varphi^{(2)}(r,z) = i\frac{\Omega\varphi_0 R}{2}\left\{\frac{H_1^{(1)}(k_+r)}{H_1^{(1)}(k_+R)}\exp\left[ik_+\left(z-\frac{h}{2}\right)\right] + \frac{H_1^{(1)}(k_-r)}{H_1^{(1)}(k_-R)}\exp\left[ik_-\left(z-\frac{h}{2}\right)\right]\right\} \qquad (2.9)$$

Region 3. ($r \leqslant R,\ z \geqslant 1/2\ h$). To make an approximate estimate of the edge effects on the velocity distribution in this region we use the following method. We analyse the effect of viscous forces on a cylindrical column of fluid over the disk. For any element of this column at height z and of thickness dz, the moment equation takes the following form

$$\pi\rho\frac{R^4}{2}dz\frac{d(v_\varphi/r)}{dt} = 2\pi\eta\int_0^R r^3\left[\left.\frac{\partial(v_\varphi/r)}{\partial z}\right|_{z+dz} - \left.\frac{\partial(v_\varphi/r)}{\partial z}\right|_z\right]dr + 2\pi\eta R^2 dz\left(\frac{\partial v_\varphi^{(2)}}{\partial r} - \frac{v_\varphi^{(2)}}{r}\right)_{r=R}$$

Assuming that in region 3 the u functions are of the form

$$u_\varphi(r,z) = rw_\varphi(z), \qquad u_r(r,z) = rw_r(z)$$

and, transforming the moment equation, after linearising with respect to w_r and w_ϕ, we get:

$$i\Omega w_\varphi + 2\omega_0 w_r = \nu\frac{d^2 w_\varphi}{dz^2} + \frac{4\nu}{R^2}\left(\frac{\partial u_\varphi^{(2)}}{\partial r} - \frac{u_\varphi^{(2)}}{r}\right)_{r=R}$$

Now combine this equation with (1.4), and we arrive at the following equations of the Meyer type (cf (8)).

$$\left[i(\Omega \mp 2\omega_0) - \nu\frac{d^2}{dz^2}\right](w_\varphi \pm iw_r) = \frac{4\nu}{R^2}\left(\frac{\partial u_\varphi^{(2)}}{\partial r} - \frac{u_\varphi^{(2)}}{r}\right)_{r=R} \qquad (2.10)$$

The solution of these heterogeneous equations with boundary conditions

$$u_r^{(3)}\left(r,\frac{h}{2}\right) = 0, \quad u_\varphi^{(3)}\left(r,\frac{h}{2}\right) = i\Omega\varphi_0 r, \quad u_r^{(3)}(r,\infty) = u_\varphi^{(3)}(r,\infty) = 0 \qquad (2.11)$$

has this form

$$u_\varphi^{(3)}(r,z) = i\frac{\Omega\varphi_0 r}{2}\left\{\left[1 + 2\frac{C_+ + C_-}{k_+^2 - k_-^2} - \frac{C_+}{ik_+}\left(z-\frac{h}{2}\right)\right]\exp\left[ik_+\left(z-\frac{h}{2}\right)\right] + \left[1 - 2\frac{C_+ + C_-}{k_+^2 + k_-^2} - \frac{C_-}{ik_-}\left(z-\frac{h}{2}\right)\right]\exp\left[ik_-\left(z-\frac{h}{2}\right)\right]\right\} \qquad (2.12)$$

in which

$$C_{\pm} = \left[\frac{d}{dr} \frac{1}{r} \frac{H_1^{(1)}(k_{\pm}r)}{H_1^{(1)}(k_{\pm}R)} \right]_{r=R} \tag{2.13}$$

It should be noted that expression (2.12) for $C_{\pm} = 0$ is in fact an exact solution of system (1.4)-(1.6) for oscillations of an infinite disk in an infinite rotating fluid. The expressions within the square brackets represent approximately calculated corrections (for) edge effects.

It is easy to see that with $z \approx 1/2\ h$, $u_{\phi}^{(3)}(R,\ z)$ transforms to $u_{\phi}^{(2)}(R,\ z)$.

In view of the fact that it is our intention to work out the moment of the viscous forces acting on the disk surface, the absence of continuity of our approximate solution in regions which are a long way from the surface is not really important. In view of this too, we will not write down expressions for the other unknown functions u_r, w_z and p_1 which do not enter the moment expression.

3. Calculation of moment of viscous forces and comparison with experimental results. The moment of forces acting on the edge and face surfaces of a disk are determined by the following formulas:

$$M^{(1)} = 2\pi\eta R^2 h \left[\frac{dv_{\varphi}^{(1)}}{dr} - \frac{v_{\varphi}^{(1)}}{r} \right]_{r=R}, \qquad M^{(3)} = 4\pi\eta \int_0^R \left[\frac{\partial v_{\varphi}^{(3)}}{\partial z} \right]_{z=1/_2h} r^2 dr \tag{3.1}$$

from which

$$M = M^{(1)} + M^{(3)} = -\frac{\pi\Omega\eta R^4}{2} \left\{ k_+ + k_- + 2\frac{C_+ + C_-}{k_+ + k_-} + \right.$$
$$\left. + \frac{C_+}{k_+} + \frac{C_-}{k_-} - 2ih(C_+ + C_-) \right\} \varphi_0 e^{i\Omega t} \equiv m\varphi_0 e^{i\Omega t} \tag{3.2}$$

Consistent with conditions (2.5) the second formula (2.1) yields

$$k_+ = -\frac{1}{\lambda_+}(1 - i), \qquad k_- = \frac{1}{\lambda_-}(\mp 1 + i) \tag{3.3}$$

The upper sign in the second formula corresponds to the case $\Omega > 2\ \omega_0$ and the lower, to the case $\Omega < 2\ \omega_0$. The depth of penetration λ_+ and λ_-, corresponding to the two possible relative directions of rotation and oscillation, will be given by

$$\lambda_{\pm} = \sqrt{\frac{2\nu}{|\Omega \pm 2\omega_0|}} \tag{3.4}$$

If the following conditions are satisfied

$$\lambda_{\pm} \ll R, \qquad h \ll R \tag{3.5}$$

expression (3.2) can be simplified using an asymptotic expansion of a Hankel function of large arguments. The expression obtained in this manner $m = (M/\phi_0)e^{+i\Omega t}$ is inserted into the known formulas

$$\Omega_0{}^2 - \Omega^2 = \frac{\mathrm{Re}\,(m)}{I}, \qquad \delta - \delta_0 \frac{\Omega_0}{\Omega} = -\frac{\pi\,\mathrm{Im}\,(m)}{I\Omega^2} \tag{3.6}$$

where Ω_0, δ_0 and Ω, δ are the frequency, and logarithmic decrement of oscillation in vacuum and in the fluid respectively and I is the moment of inertia of the disk. It is assumed that damping is weak

$$\delta \ll 1$$

For the case $\Omega > 2\,\omega_0$ we have

$$2I\,(\Omega_0{}^2 - \Omega^2) = \pi\eta R^4\,\Omega\left(\frac{1}{\lambda_+} + \frac{1}{\lambda_-}\right)\left(1 + \frac{2h}{R}\right) \tag{3.8}$$

$$2I\Omega\left(\delta - \delta_0\frac{\Omega_0}{\Omega}\right) = \pi^2\eta R^4\left(\frac{1}{\lambda_+} + \frac{1}{\lambda_-}\right)\left(1 + \frac{2h}{R} + \frac{4}{R}\,\frac{\lambda_+\lambda_-}{\lambda_+ + \lambda_-}\right) \tag{3.9}$$

The latter formula transforms to the familiar one for a fluid at rest incorporating the Landau correction for the case $\omega_0 = 0$ [2,3,8].

For the case $\Omega < 2\,\omega_0$ formula (3.9) remains valid, and instead of (3.8) we have

$$2I\,(\Omega_0{}^2 - \Omega^2) = \pi\eta\,R^4\Omega\left(\frac{1}{\lambda_+} - \frac{1}{\lambda_-}\right)\left(1 + \frac{2h}{R}\right) \tag{3.10}$$

Formula (3.9) has been confirmed experimentally by D.S. Tsakadze and K.B. Mesoyed, when studying the oscillations of a "heavy" disk ($\Omega \approx \Omega_0$) in distilled water.

In Fig. 2 we can see complete agreement between formula (3.9) (full lines) and experiment. In the same graph we give results of calculation (broken line) without allowing for the correction

$$\left(1 + \frac{2h}{R} + \frac{4}{R}\,\frac{\lambda_+\lambda_-}{\lambda_+ + \lambda_-}\right)$$

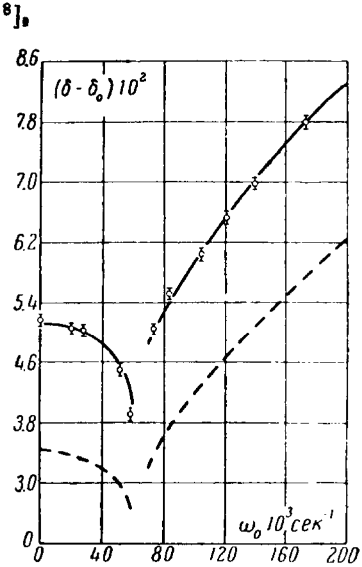

Fig. 2.

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